Conjunctive networks

Complexity of limit cycle problems with different schedules

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Boolean networks and interaction digraph

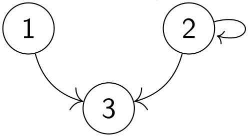
Boolean networks:

- Global function: $f : \{0,1\}^n \to \{0,1\}^n$.
- Local functions: $f_1, \dots, f_n : \{0,1\}^n \to \{0,1\}$.
- $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$

Local functions:

- $f_1 : x \mapsto 1$.
- $f_2 : x \mapsto x_2$.
- $f_3 : x \mapsto x_1 \vee x_2$.

Interaction digraph D_f:



$\mathcal{N}^{\mathrm{in}}(1) = \emptyset, \mathcal{N}^{\mathrm{in}}(2) = \{x_2\}$ and $\mathcal{N}^{\mathrm{in}}(3) = \{x_1, x_2\}$.

Conjunctive networks

A conjunctive networks $f : \{0,1\}^n \to \{0,1\}^n$:

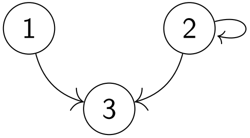
$$\forall j \in [n], f_j : x \mapsto \bigvee_{i \in \mathcal{N}^{\text{in}}(j)} x_i$$

(if $\mathcal{N}^{\text{in}}(j) = \emptyset,\ f_j(x) = 0$).

Local functions:

- $f_1 : x \mapsto 0$.
- $f_2 : x \mapsto x_2$.
- $f_3 : x \mapsto x_1 \vee x_2$.

Interaction digraph D_f:



Limit cycles

$x \in \{0,1\}^n$ is in a limit cycle of f of length k if

- $\forall 1 \leq q < k, f^q(x) \neq x$, and
- $\forall f^k(x) = x$.

Notations:

- $\phi_k(f)$: number of limit cycles of length k of f: .
- $\Phi_k(f)$: configurations in a limit cycle of length k of f:

We have $\phi_k(f) = |\Phi_k(f)|/k$.

Decision problems

For any constant k, we define the following problems.

Definition: k-Parallel Limit Cycle problem (k-PLC)

Given a conjunctive network f, does $\phi_k(f) \geq 1$?

Definition: k-Block-sequential Limit Cycle problem (k-BLC)

Given a conjunctive network f, does there exist a block-sequential schedule w such that $\phi_k(f^w) \geq 1$?

Definition: k-Sequential Limit Cycle problem (k-SLC)

Given a conjunctive network f, does there exist a sequential schedule w such that $\phi_k(f^w) \geq 1$?

Remark

All this problems are trivial for $k = 1$.

k-PLC

Theorem

For all $k \geq 2$, The k-PLC problem can be resolved in polynomial time.

See: Disjunctive networks and update schedules, Eric Goles and Mathilde Noual, 2011.

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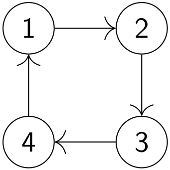
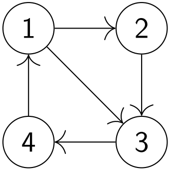
When D_f is strongly connected, it is equivalent to know if there exists a function $c : [n] \to [0, k-1]$ such that for all $i, j \in [n]$, $i \in \mathcal{N}^{\text{in}}(j) \implies c(j) = c(i) + 1 \mod k$.

Example: 2-PLC problem for the two following interaction digraphs.





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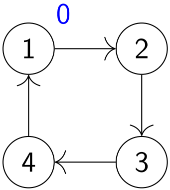

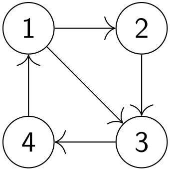

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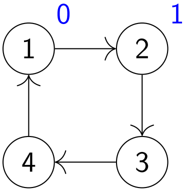

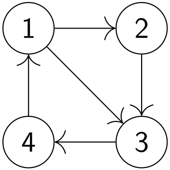

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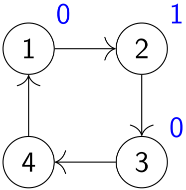

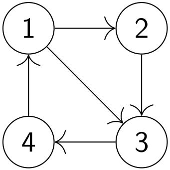

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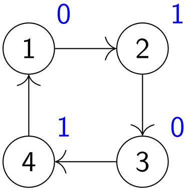

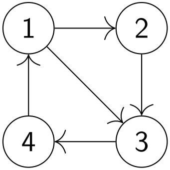

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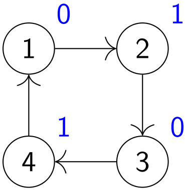

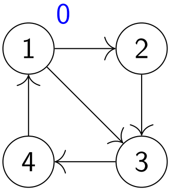

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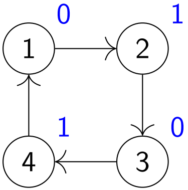

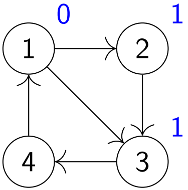

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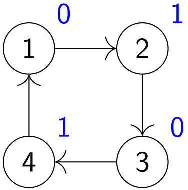
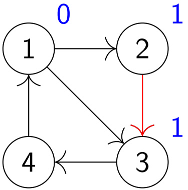
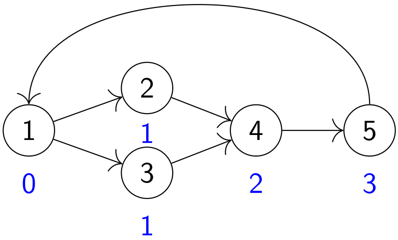




k-PLC

Proof of: c exists $\Rightarrow \phi_k(f) \geq 1$.



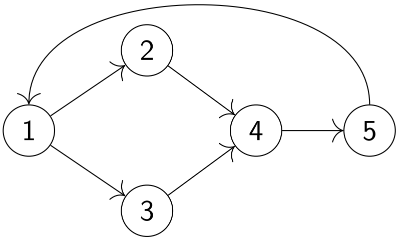
$x^{(t)} : \forall i \in [n], c(i) = t \iff x_i^{(t)} = 1.$

$$x_i^{(0)} \quad \xrightarrow{f} x_i^{(1)} \quad \xrightarrow{f} x_i^{(2)} \quad \xrightarrow{f} x_i^{(3)} \quad \xrightarrow{f} x_i^{(0)}$$

$$10000 \quad \xrightarrow{f} 01100 \quad \xrightarrow{f} 00010 \quad \xrightarrow{f} 00001 \quad \xrightarrow{f} 10000$$

k-PLC

Proof of: $\phi_k(f) \geq 1 \Rightarrow c$ exists .
Let $x \in \Phi_k(f)$. Periodic trace: for all $i \in [n]$,
$p^i(x) = (x_i, f_i(x), f_i^2(x)), \ldots, (f_i^{k-1}(x)))$
In this example, consider that $i \in [2]$ with the maximum 1 in its periodic trace is 2.



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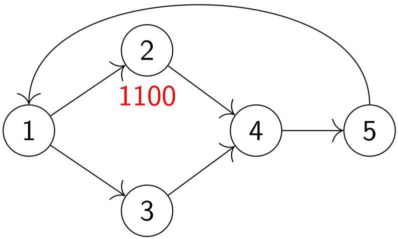

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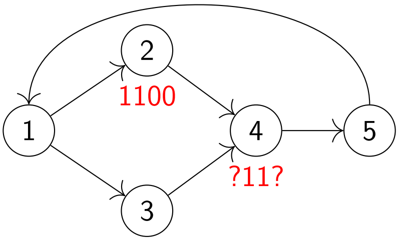

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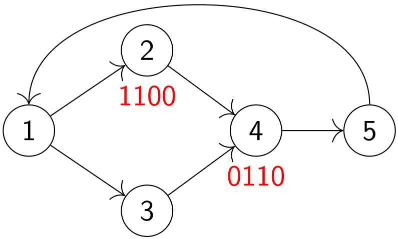

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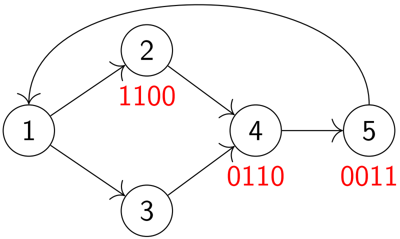

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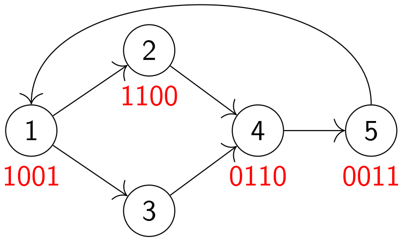

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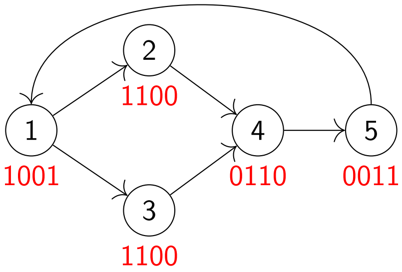

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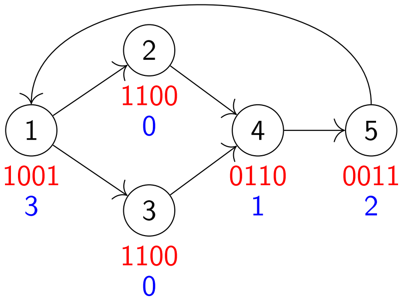

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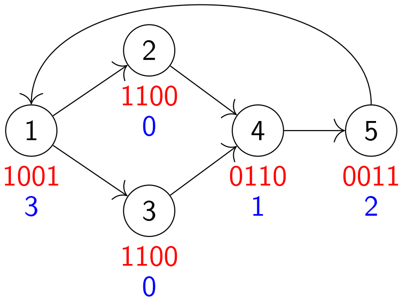


Is to possible to have $0 \to 2$ for example?
$\implies$ No, because otherwise the period would not be minimum.

k-BLC et k-SLC

Lemma [Eric Goles and Mathilde Noual, 2011]

For any disjunctive network f, there exists a block-sequential update schedule w such that f^w only has fixed points.

Theorem

The k-BLC et k-SLC problems are NP-complete.

To resolve these two problems when D_f is strongly connected, it is sufficient to execute the following non-deterministic polynomial time algorithm.

- Chose a (block)-sequential update schedule w.
- Chose a configuration $x \in \{0,1\}^n$.
- Verify that $(f^w)^k(x) = x$ and that for all $q \in [1, k-1]$, $(f^w)^q(x) \neq x$.

This problems are thus in NP.

2-BLC and 2-SLC

When D_f is strongly connected, it is equivalent to find a update digraph and a function $c : [n] \to [0, k-1]$ such that

- $(i) \xrightarrow{\oplus} (j) \implies c(j) = c(i) + 1 \mod k$.
- $(i) \xrightarrow{\ominus} (j) \implies c(j) = c(i)$.

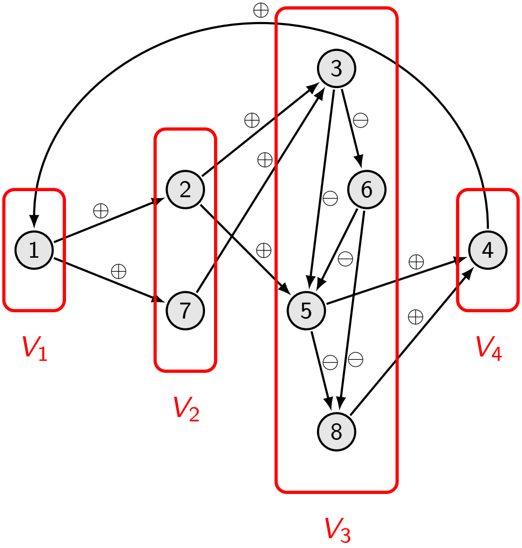
Lemma

An upgrade digraph corresponds to a **sequential** update schedule if when we reverse every negative arcs, the digraph becomes acyclic.

Lemma

An upgrade digraph corresponds to a **block-sequential** update schedule if when we reverse every negative arcs, the only remaining cycles are only composed of positive arcs.

2-BLC and 2-SLC



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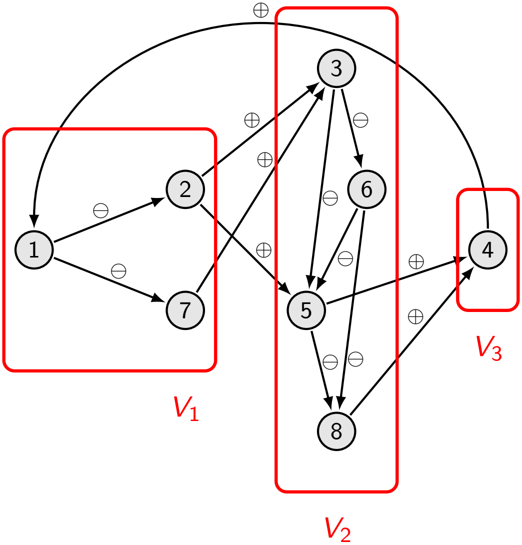

Reduction of 2-BLC and 2-SLC from 3-SAT

3-SAT problem: $(\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3)$

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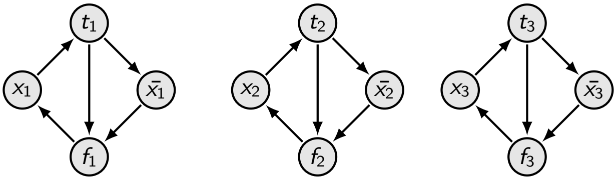

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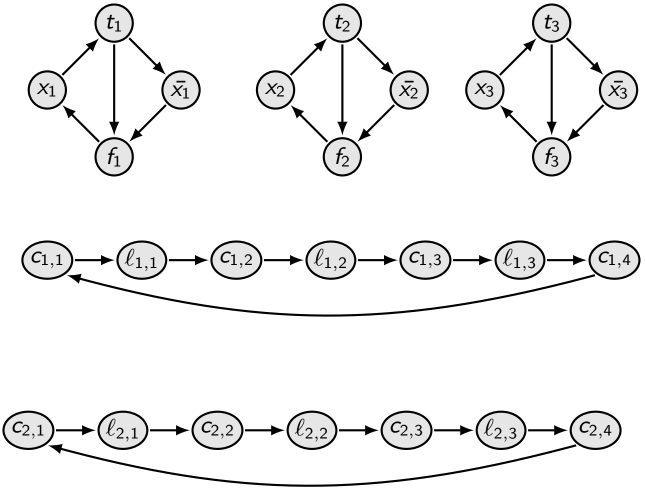

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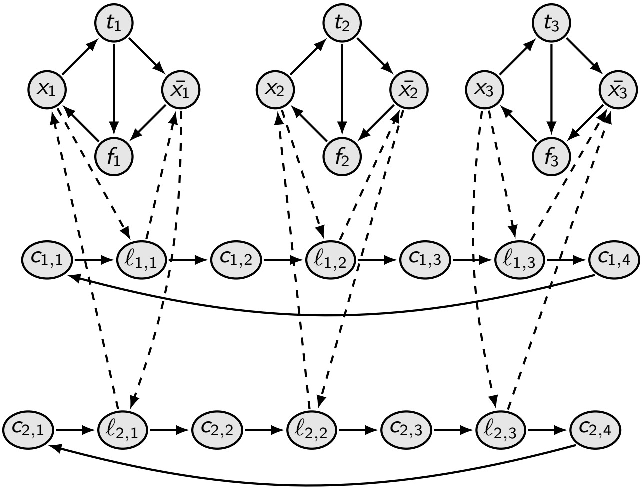

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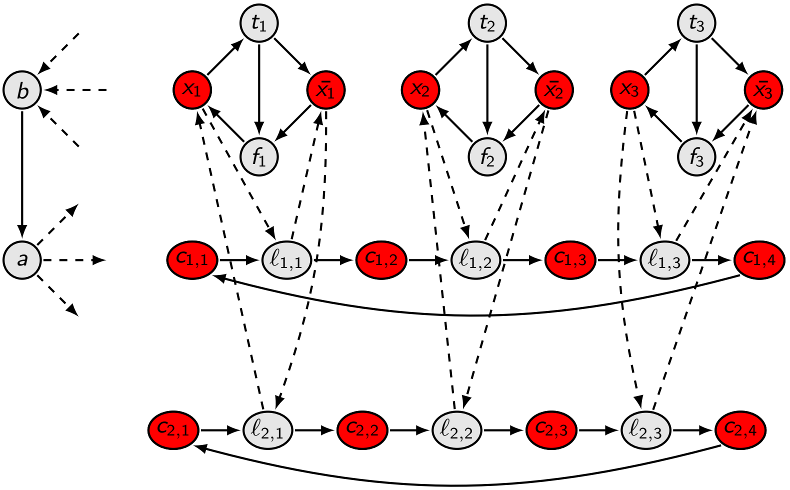

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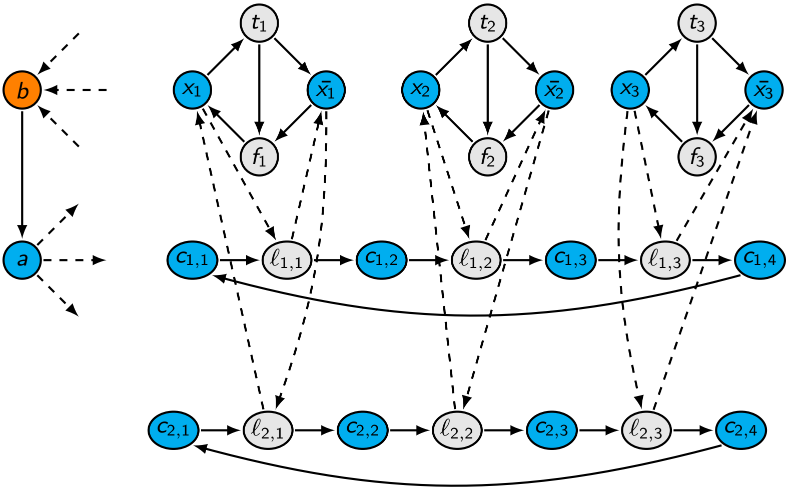

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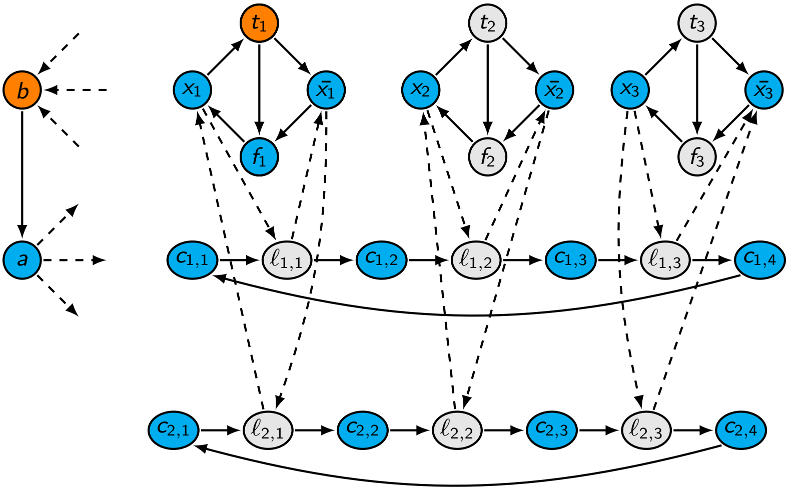

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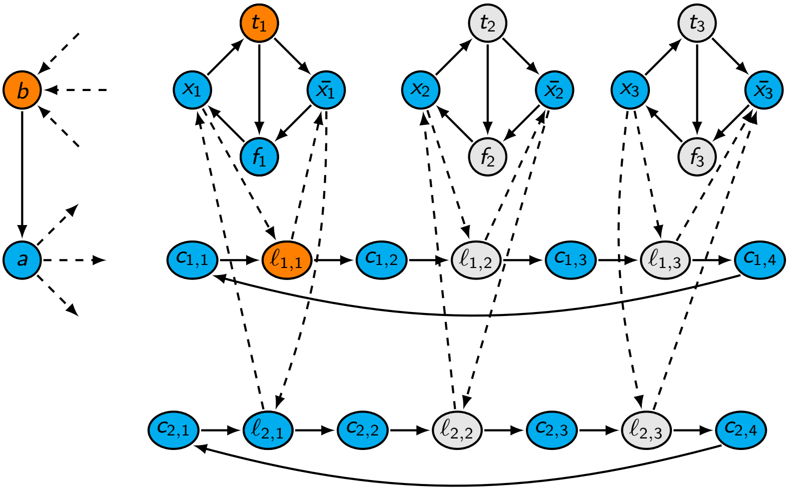

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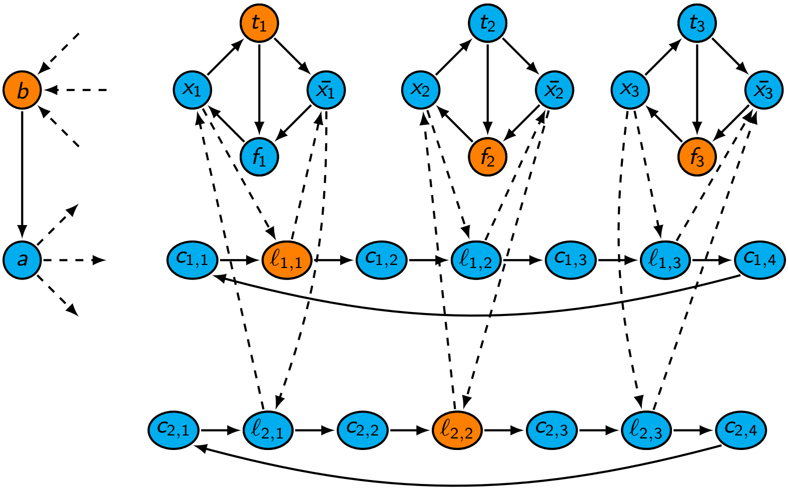
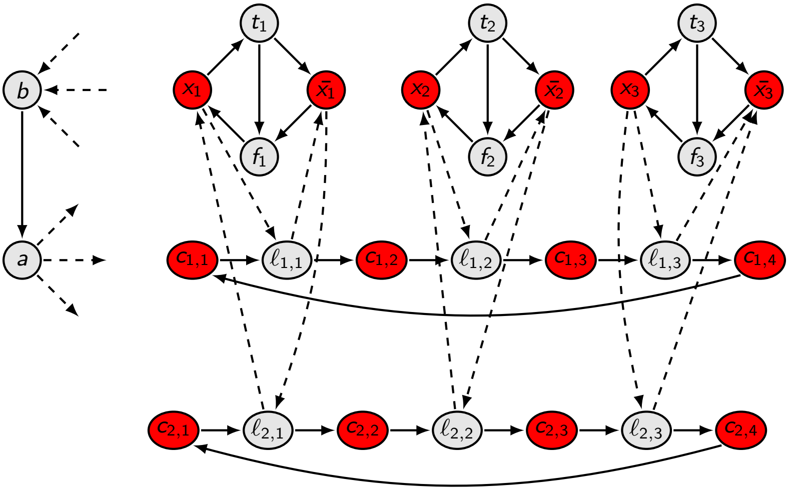


Reduction of 2-BLC and 2-SLC from 3-SAT

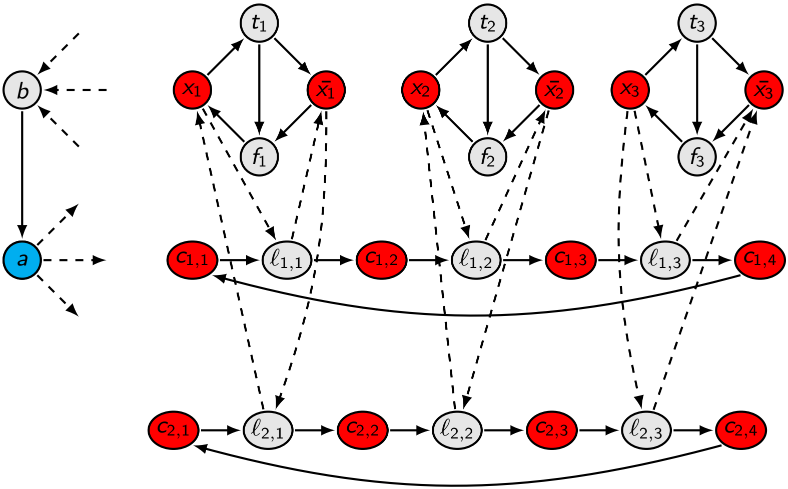
Lemma

Let f be a conjunctive network and k an integer. Then, $\phi_k(f^{(w_1 \dots w_{n-1} w_n)}) = \phi_k(f^{(w_n w_1 \dots w_{n-1})})$.

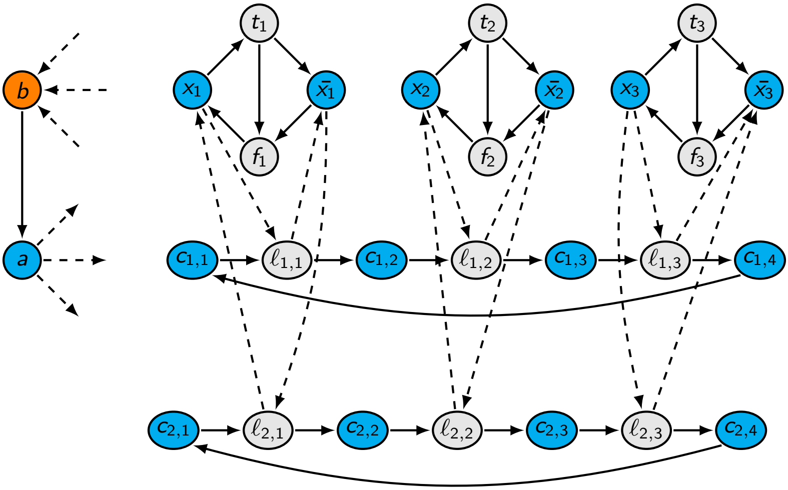
3-SAT problem: $(\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3)$



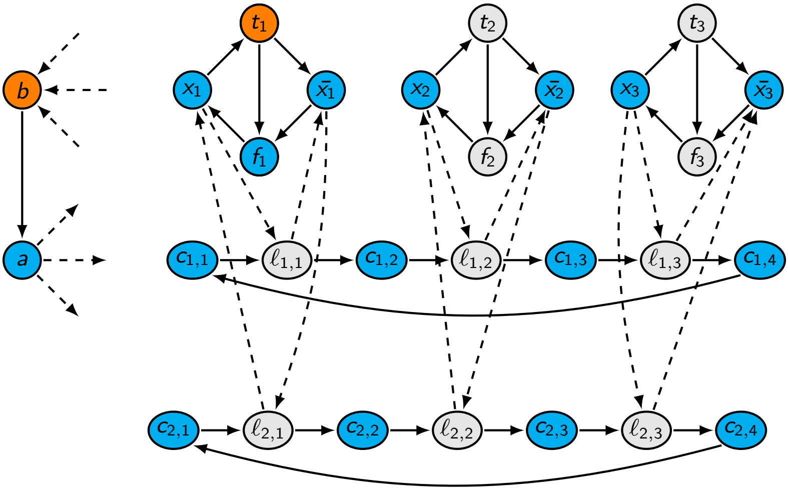
3-SAT problem: $(\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3)$



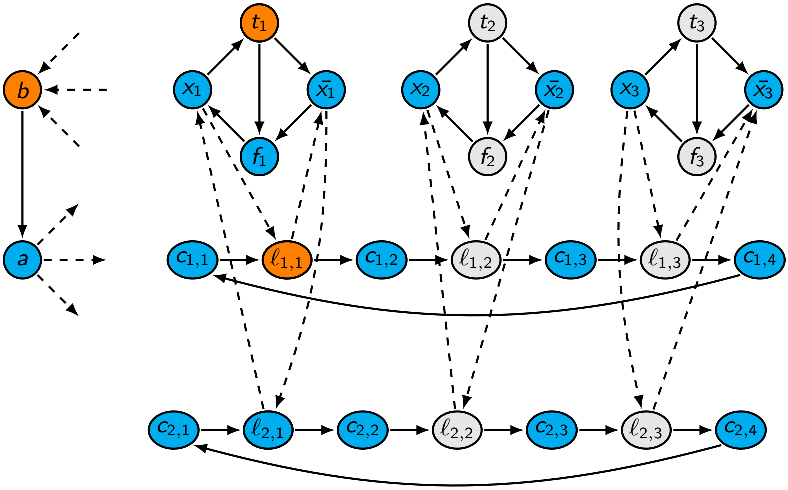
3-SAT problem: $(\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3)$



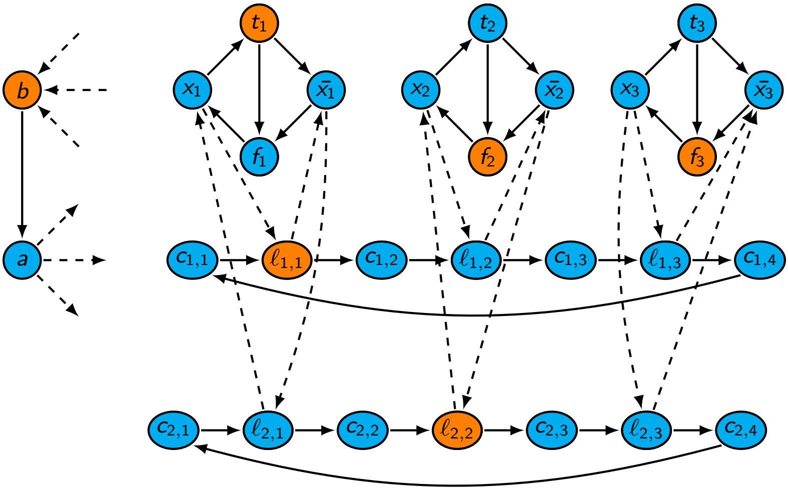
3-SAT problem: $(\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3)$



3-SAT problem: $(\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3)$



3-SAT problem: $(\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\neg\lambda_1 \vee \neg\lambda_2 \vee \lambda_3)$



Conclusion

Results:

- The k-PLC problem can be resolved in polynomial time.
- The k-BLC and k-SLC problems are NP-complete for any $k \geq 2$.

Ongoing:

- Not strongly connected.

Future works:

- Does the complexity change when k is not a constant but a problem parameter?
- The problem of computing $\phi_k(f^w)$ is it difficult?