# Fixed points and feedback cycles in Boolean networks

Adrien Richard

CNRS & Université Côte d'Azur, France

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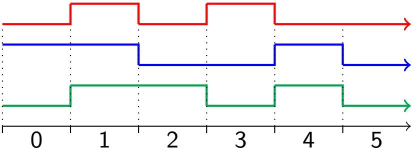

Dynamic

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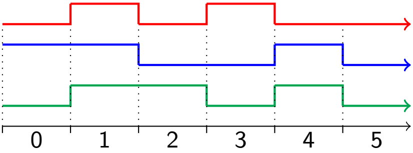
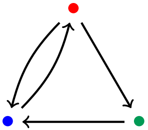


Dynamic



Interaction graph

A **Boolean network (BN)** is a **discrete dynamical system** containing a **finite number of binary variables** which evolve, in a **discrete time** and through **mutual interactions**, according to a **fixed law**.

$\hookrightarrow$ The **variables**/**components** are indexed from $1$ to $n$.

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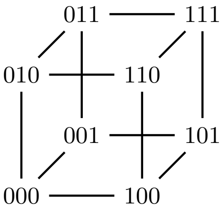
**Example** with $n = 3$

$$\{0,1\}^3 = \begin{cases} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{cases}$$



Hypercube

A **Boolean network** with $n$ components is a function

$$\begin{aligned} f : \{0,1\}^n &\to \{0,1\}^n \\ x=(x_1,\dots,x_n) &\mapsto f(x)=(f_1(x),\dots,f_n(x)) \end{aligned}$$

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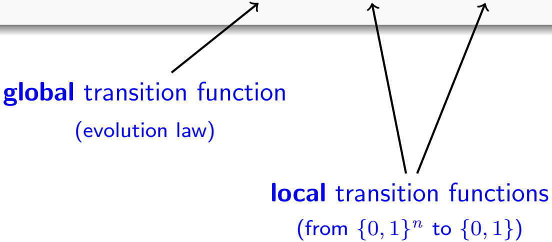

**global** transition function

(evolution law)

**local** transition functions

(from $\{0,1\}^n$ to $\{0,1\}$)

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The **dynamic** is given by the successive iterations of $f$ :

$$x \to f(x) \to f^2(x) \to f^3(x) \to \cdots$$

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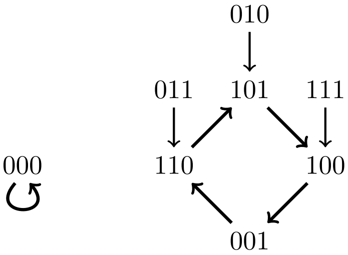
A **fixed point** is a configuration $x$ such that $x = f(x)$.

**fixed points = stable states**

# **Example 1** with $n = 3$

$$\begin{cases} f_1(x) & = x_2 \vee x_3 \\ f_2(x) & = \overline{x_1} \wedge \overline{x_3} \\ f_3(x) & = \overline{x_3} \wedge (x_1 \vee x_2) \end{cases}$$

| $x$ | $f(x)$ |
|---|---|
| 000 | 000 |
| 001 | 110 |
| 010 | 101 |
| 011 | 110 |
| 100 | 001 |
| 101 | 100 |
| 110 | 101 |
| 111 | 100 |



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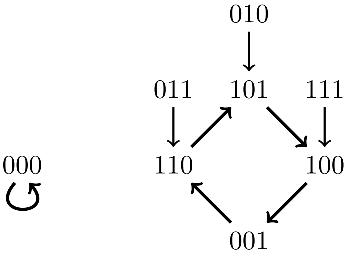

**Exercise :** What is the nb of BNs with $n$ components ?

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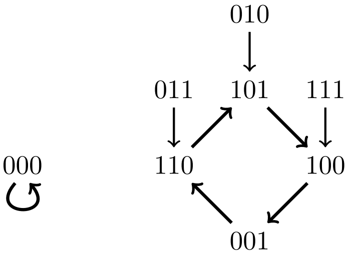

**Exercise :** What is the nb of BNs with $n$ components ? → $\boldsymbol{(2^n)^{(2^n)} = 2^{n2^n}}$

The **signed interaction graph** of $f$ is the **signed digraph** $G$ defined by :

- the set of vertices is $\{1,\dots,n\}$
- there is a positive arc $\boldsymbol{j \to i}$ if there is $x\in\{0,1\}^n$ such that

$$\begin{aligned}f_i(x_1,\dots,x_{j-1},\mathbf{0},x_{j+1},\dots,x_n)&=\mathbf{0}\\ f_i(x_1,\dots,x_{j-1},\mathbf{1},x_{j+1},\dots,x_n)&=\mathbf{1}\end{aligned}$$

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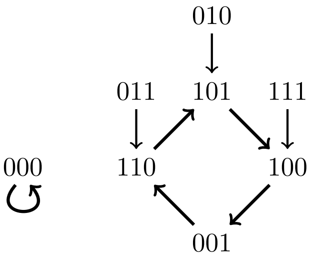
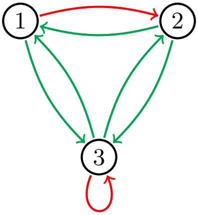
### Dynamic



### Interaction graph



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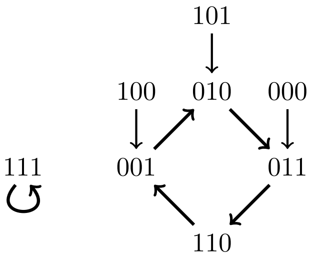

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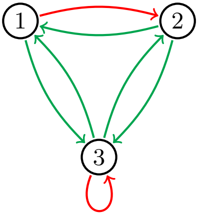

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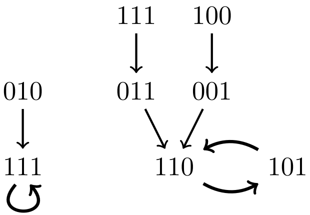

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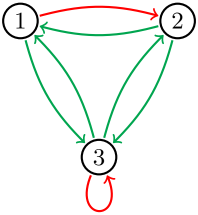

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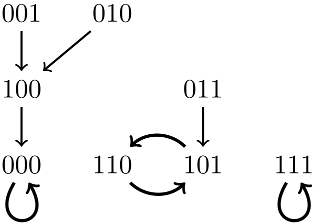
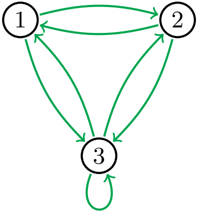
|---|---|
| 000 | 000 |
| 001 | 100 |
| 010 | 100 |
| 011 | 101 |
| 100 | 000 |
| 101 | 110 |
| 110 | 101 |
| 111 | 111 |

Dynamic



Interaction graph



## Example 5

$$\begin{cases} f_1(x) & = x_2 + x_3 \\ f_2(x) & = x_1 \wedge x_3 \\ f_3(x) & = x_3 \wedge (x_1 \vee x_2) \end{cases}$$

| $x$ | $f(x)$ |
|---|---|
| 000 | 000 |
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| 011 | 001 |
| 100 | 000 |
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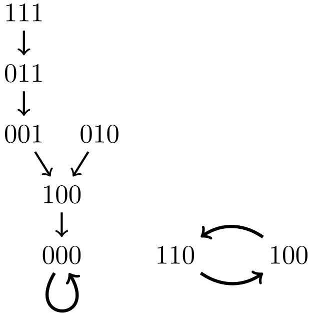
Dynamic



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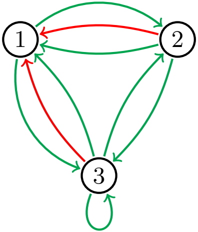

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|---|---|
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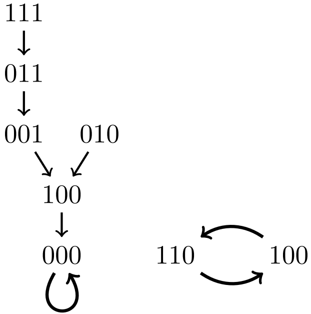

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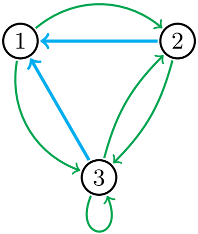

# Example 6

$$\begin{cases} f_1(x) &= x_2 + x_3 \\ f_2(x) &= x_3 + x_1 \\ f_3(x) &= x_1 + x_2 \end{cases}$$

| $x$ | $f(x)$ |
|---|---|
| 000 | 000 |
| 001 | 110 |
| 010 | 101 |
| 011 | 011 |
| 100 | 011 |
| 101 | 101 |
| 110 | 110 |
| 111 | 000 |

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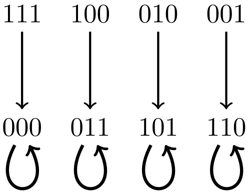
111 100 010 001

000 011 101 110

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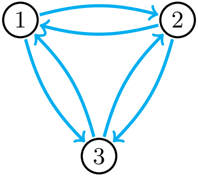
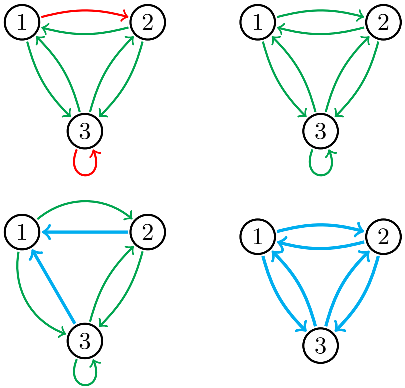
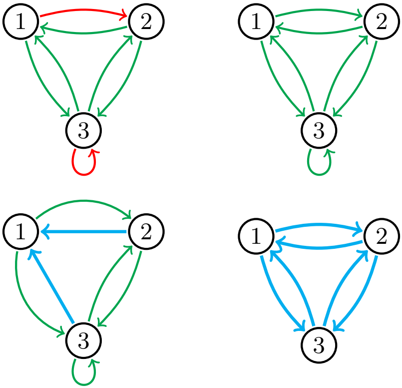




**Exercise :** What is the nb of signed digraphs with $n$ vertices ?

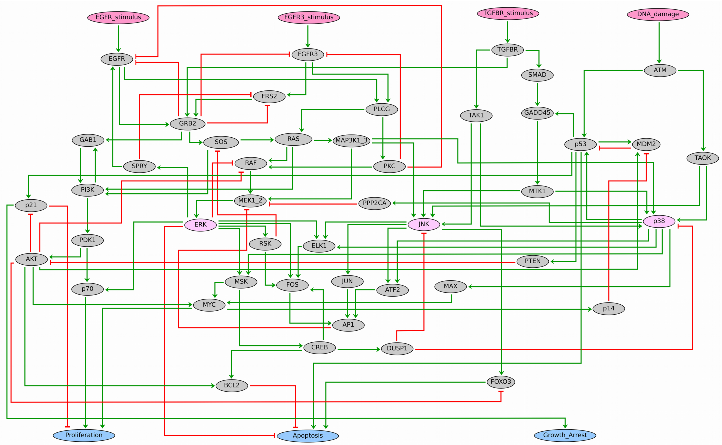


**Exercise :** What is the nb of signed digraphs with $n$ vertices ? $\rightarrow \boldsymbol{4^{n^2}}$.

**Many applications**, in particular :

- Neural networks [McCulloch & Pitts 1943]
- **Gene networks** [Kauffman 1969, Thomas 1973]

# In the context of gene networks, the first reliable informations often concern the interaction graph



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**Question**

1. *What can be said on the* **dynamic** *of a Boolean network according to its* **interaction graph** *only ?*

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**Difficult question**

$\hookrightarrow$ the nb of BNs on a given interaction graph $G$ is (generally) **HUGE**.

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$\hookrightarrow$ the nb of BNs on a *random* interaction graph $G$ is **doubly exponential**.

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Number of fixed points in the gene network of a multicellular organism ≈ Number of cellular types in the organism

### Definitions

$$\begin{aligned}\mathbf{max}(\boldsymbol{G}) &:= \textbf{maximum number of fixed points} \text{ in a BN on } G\\ \mathbf{min}(\boldsymbol{G}) &:= \textbf{minimum number of fixed points} \text{ in a BN on } G\end{aligned}$$

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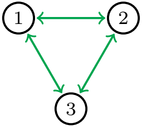
$K_3^+$



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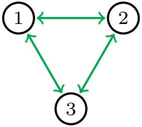

There are $8$ possibles BNs on $K_3^+$, since

$$f_1(x) = x_2 \wedge x_3 \quad \text{or} \quad f_1(x) = x_3 \vee x_3$$

$$f_2(x) = x_1 \wedge x_3 \quad \text{or} \quad f_2(x) = x_1 \vee x_3$$

$$f_3(x) = x_1 \wedge x_2 \quad \text{or} \quad f_3(x) = x_1 \vee x_2$$

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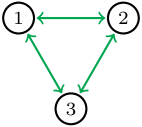

| $x$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ |
|---|---|---|---|---|---|---|---|---|
| 000 | **000** | **000** | **000** | **000** | **000** | **000** | **000** | **000** |
| 001 | 000 | 100 | 010 | 110 | 000 | 100 | 010 | 110 |
| 010 | 000 | 100 | 000 | 100 | 001 | 101 | 001 | 101 |
| 011 | 100 | 100 | 110 | 110 | 101 | 101 | 111 | 111 |
| 100 | 000 | 000 | 010 | 010 | 001 | 001 | 011 | 011 |
| 101 | 010 | 110 | 010 | 110 | 011 | 111 | 011 | 111 |
| 110 | 001 | 101 | 011 | 111 | 001 | 101 | 011 | 111 |
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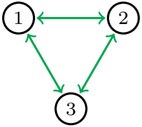

8 possible BNs

$$\max(K_3^+) = 2$$
$$\min(K_3^+) = 2$$

| $x$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ |
|---|---|---|---|---|---|---|---|---|
| 000 | **000** | **000** | **000** | **000** | **000** | **000** | **000** | **000** |
| 001 | 000 | 100 | 010 | 110 | 000 | 100 | 010 | 110 |
| 010 | 000 | 100 | 000 | 100 | 001 | 101 | 001 | 101 |
| 011 | 100 | 100 | 110 | 110 | 101 | 101 | 111 | 111 |
| 100 | 000 | 000 | 010 | 010 | 001 | 001 | 011 | 011 |
| 101 | 010 | 110 | 010 | 110 | 011 | 111 | 011 | 111 |
| 110 | 001 | 101 | 011 | 111 | 001 | 101 | 011 | 111 |
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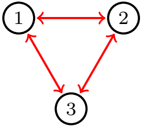
$\mathbf{min}(G)$ := **minimum number of fixed points** in a BN on $G$

$K_3^-$

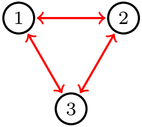


**Definitions**

$$\mathbf{max}(G) \;:=\; \textbf{maximum number of fixed points} \text{ in a BN on } G$$

$$\mathbf{min}(G) \;:=\; \textbf{minimum number of fixed points} \text{ in a BN on } G$$

$K_3^-$



There are $8$ possible BNs on $K_3^-$, since

$$f_1(x) = \overline{x_2} \wedge \overline{x_3} \quad \text{or} \quad f_1(x) = \overline{x_3} \vee \overline{x_3}$$

$$f_2(x) = \overline{x_1} \wedge \overline{x_3} \quad \text{or} \quad f_2(x) = \overline{x_1} \vee \overline{x_3}$$

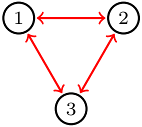
$$f_3(x) = \overline{x_1} \wedge \overline{x_2} \quad \text{or} \quad f_3(x) = \overline{x_1} \vee \overline{x_2}$$

## Definitions

$\max(G)$ := **maximum number of fixed points** in a BN on $G$

$\min(G)$ := **minimum number of fixed points** in a BN on $G$

$K_3^-$



| $x$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ |
|---|---|---|---|---|---|---|---|---|
| 000 | 111 | 111 | 111 | 111 | 111 | 111 | 111 | 111 |
| 001 | **001** | 101 | 011 | 111 | **001** | 101 | 011 | 111 |
| 010 | **010** | 110 | **010** | 110 | 011 | 111 | 011 | 111 |
| 011 | 000 | 000 | 010 | 010 | 001 | 001 | **011** | **011** |
| 100 | **100** | **100** | 110 | 110 | 101 | 101 | 111 | 111 |
| 101 | 000 | 100 | 000 | 100 | 001 | **101** | 001 | **101** |
| 110 | 000 | 100 | 010 | **110** | 000 | 100 | 010 | **110** |
| 111 | 000 | 000 | 000 | 000 | 000 | 000 | 000 | 000 |

**Definitions**

$\max(G)$ := **maximum number of fixed points** in a BN on $G$

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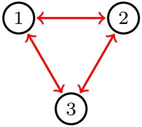
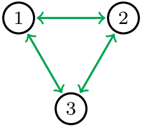
$K_3^-$



8 possible BNs

$$\max(K_3^-) = 3$$
$$\min(K_3^-) = 1$$

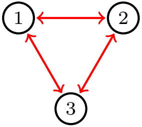
| $x$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ | $f(x)$ |
|---|---|---|---|---|---|---|---|---|
| 000 | 111 | 111 | 111 | 111 | 111 | 111 | 111 | 111 |
| 001 | **001** | 101 | 011 | 111 | **001** | 101 | 011 | 111 |
| 010 | **010** | 110 | **010** | 110 | 011 | 111 | 011 | 111 |
| 011 | 000 | 000 | 010 | 010 | 001 | 001 | **011** | **011** |
| 100 | **100** | **100** | 110 | 110 | 101 | 101 | 111 | 111 |
| 101 | 000 | 100 | 000 | 100 | 001 | **101** | 001 | **101** |
| 110 | 000 | 100 | 010 | **110** | 000 | 100 | 010 | **110** |
| 111 | 000 | 000 | 000 | 000 | 000 | 000 | 000 | 000 |



8 BNs

$\max(K_3^+) = 2$

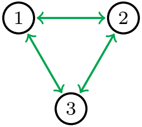
$\min(K_3^+) = 2$



8 BNs

$\max(K_3^-) = 3$

$\min(K_3^-) = 1$



$8$ BNs

$\max(K_3^+) = 2$

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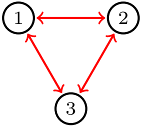

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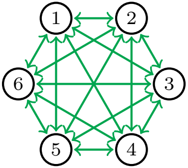
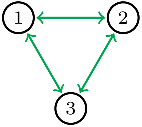


$\sim 10^{41}$ BNs

$4 \leq \max(K_6^+) \leq 16$

$\min(K_6^+) = 2$



$8$ BNs

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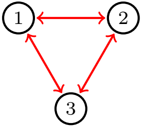
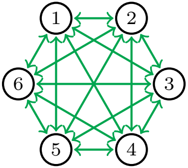


$8$ BNs

$\max(K_3^-) = 3$

$\min(K_3^-) = 1$



$\sim 10^{41}$ BNs

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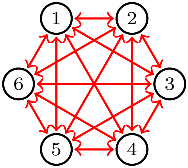

$\sim 10^{41}$ BNs

$\max(K_6^-) = 20$

$\min(K_6^+) = 0$

## Outline

1. Absence of cycle
2. Positive and negative cycles
3. Absence of positive/negative cycle
4. Positive feedback bound
5. Positive and negative cliques
6. The monotone case
7. Conclusion

# Outline

1. **Absence of cycle**
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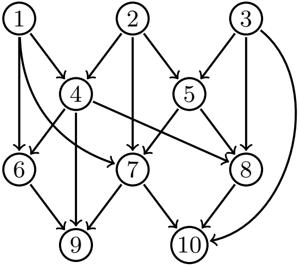
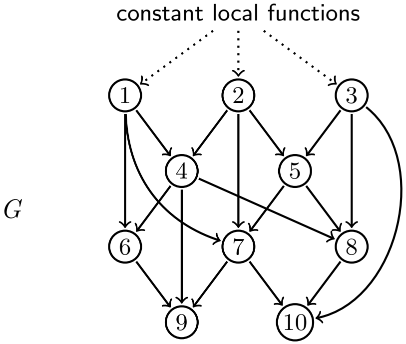
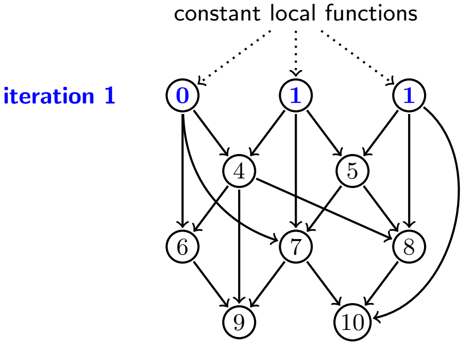
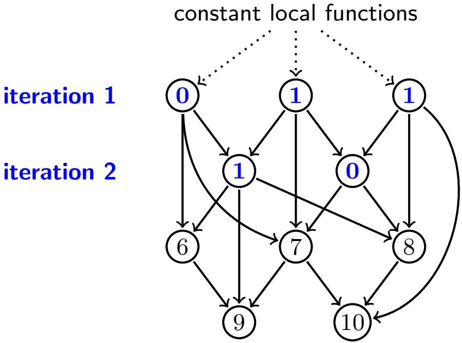
$G$




constant local functions
G
1
2
3
4
5
6
7
8
9
10

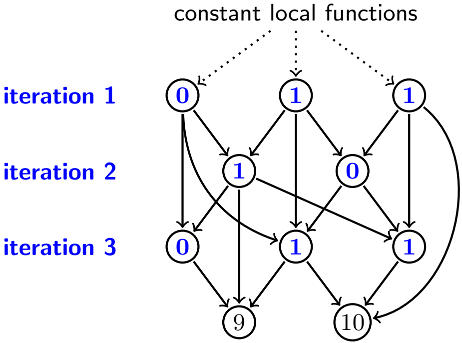

constant local functions
iteration 1
0
1
1
4
5
6
7
8
9
10



constant local functions

**iteration 1**

**iteration 2**

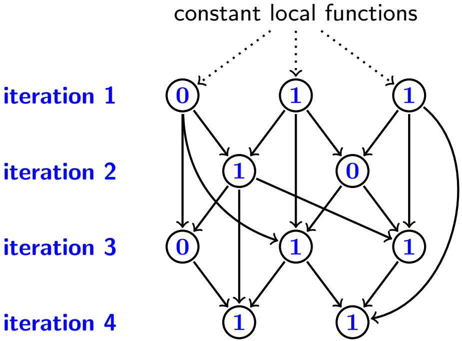


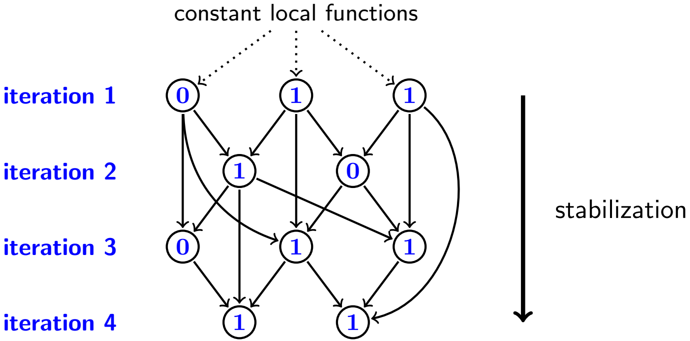
constant local functions

**iteration 1**

**iteration 2**

**iteration 3**


constant local functions
iteration 1
iteration 2
iteration 3
iteration 4
0
1
1
1
0
0
1
1
1
1



constant local functions

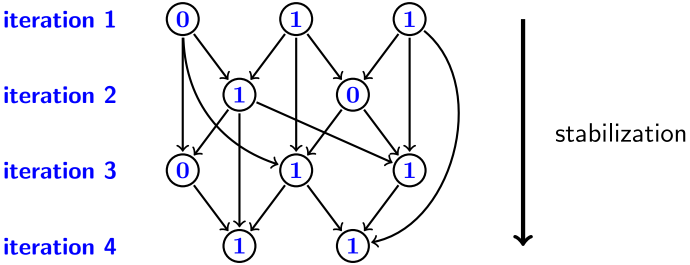
**iteration 1**

**iteration 2**

**iteration 3**

**iteration 4**

stabilization



**iteration 1**

**iteration 2**

**iteration 3**

**iteration 4**

stabilization

**Theorem** [Robert, 1980]

*If $G$ is acyclic then $f^n$ is a constant function, thus*

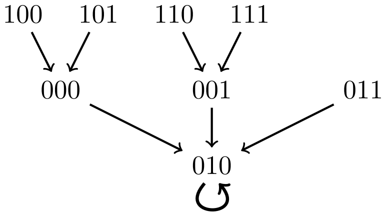
$$\min(G) = \max(G) = 1.$$

## Example 1

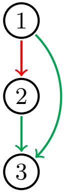
$$\begin{cases} f_1(x) & = 0 \\ f_2(x) & = \overline{x_1} \\ f_3(x) & = x_1 \wedge x_2 \end{cases}$$

| $x$ | $f(x)$ |
|---|---|
| 000 | 010 |
| 001 | 010 |
| 010 | 010 |
| 011 | 010 |
| 100 | 000 |
| 101 | 000 |
| 110 | 001 |
| 111 | 001 |

Dynamic



Interaction graph

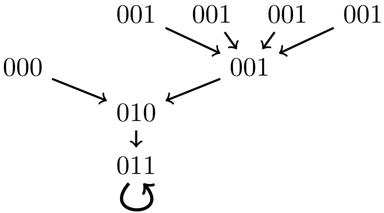


## Example 2

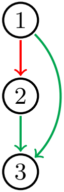
$$\begin{cases} f_1(x) & = 0 \\ f_2(x) & = \overline{x_1} \\ f_3(x) & = x_1 \vee x_2 \end{cases}$$

| $x$ | $f(x)$ |
|---|---|
| 000 | 010 |
| 001 | 010 |
| 010 | 011 |
| 011 | 011 |
| 100 | 001 |
| 101 | 001 |
| 110 | 001 |
| 111 | 001 |

Dynamic



Interaction graph



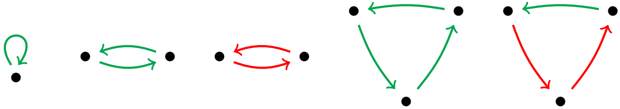
**François Robert** [1980]

$$\text{no cycle} \quad \Rightarrow \quad \text{“simple” dynamic}$$

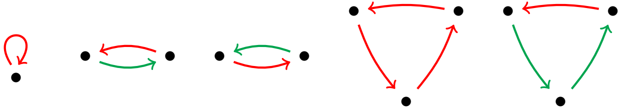
$$\text{“complexe” dynamic} \quad \Rightarrow \quad \text{cycles}$$

**René Thomas** [1981] : two type of cycles, **positive** and **negative**.

1. **Positive cycle** : **even** number of **negative** arcs



2. **Negative cycle** : **even** number of **negative** arcs



# Outline

1. Absence of cycle
2. **Positive and negative cycles**
3. Absence of positive/negative cycle
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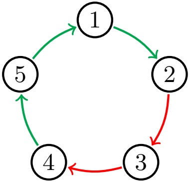
7. Conclusion

In a cycle, each vertex $i$ has a unique in-neighbor $j$, and

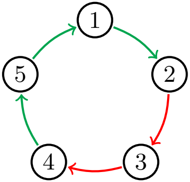
$$f_i(x) = \begin{cases} x_j & \text{if } j \to i \text{ is positive} \\ \overline{x_j} & \text{if } j \to i \text{ is negative} \end{cases}$$

**Example**



$$f_1(x) = x_5$$
$$f_2(x) = x_1$$
$$f_3(x) = \overline{x_2}$$
$$f_4(x) = \overline{x_3}$$
$$f_5(x) = x_4$$

## Fixed points for a positive cycle



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$$x = f(x) \quad \iff \quad \begin{cases} x_1 = x_5 \\ x_2 = x_1 \\ x_3 = \overline{x_2} \\ x_4 = \overline{x_3} \\ x_5 = x_4 \end{cases}$$

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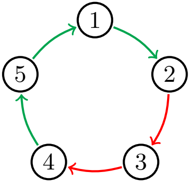

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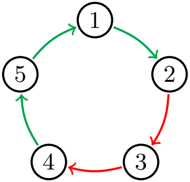

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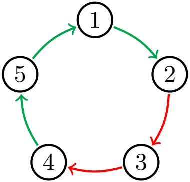
$$f_5(x) = x_4$$

$$x = f(x) \quad \Longleftrightarrow \quad \begin{cases} x_1 = x_5 \\ x_2 = x_1 \\ x_3 = \overline{x_2} = \overline{x_1} \\ x_4 = \overline{x_3} = x_1 \\ x_5 = x_4 \end{cases}$$

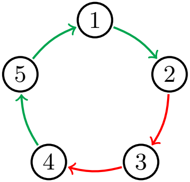
## Fixed points for a positive cycle



$$
\begin{aligned}
f_1(x) &= x_5\\
f_2(x) &= x_1\\
f_3(x) &= \overline{x_2}\\
f_4(x) &= \overline{x_3}\\
f_5(x) &= x_4
\end{aligned}
$$

$$
x = f(x) \iff \begin{cases}
x_1 = x_5\\
x_2 = x_1\\
x_3 = \overline{x_2} = \overline{x_1}\\
x_4 = \overline{x_3} = x_1\\
x_5 = x_4 = x_1
\end{cases}
$$

## Fixed points for a positive cycle



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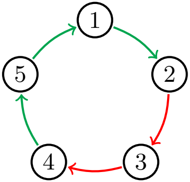
$$x = f(x) \iff \begin{cases} x_1 = x_5 \\ x_2 = x_1 \\ x_3 = \overline{x_2} = \overline{x_1} \\ x_4 = \overline{x_3} = x_1 \\ x_5 = x_4 = x_1 \end{cases} \iff x = (x_1, x_1, \overline{x_1}, x_1, x_1)$$

## Fixed points for a positive cycle



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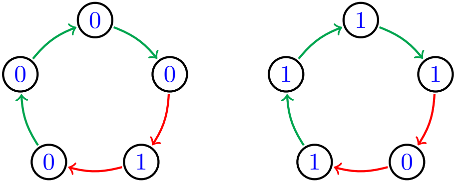
There are exactly two fixed points : **00100** and **11011**.

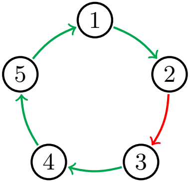
## Fixed points for a positive cycle



$$x = f(x) \iff \begin{cases} x_1 = x_5 \\ x_2 = x_1 \\ x_3 = \overline{x_2} = \overline{x_1} \\ x_4 = \overline{x_3} = x_1 \\ x_5 = x_4 = x_1 \end{cases} \iff x = (x_1, x_1, \overline{x_1}, x_1, x_1)$$

There are exactly two fixed points : **00100** and **11011**.

# Fixed points for a negative cycle



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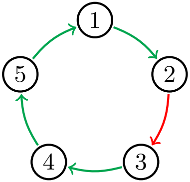
$$x = f(x) \quad \Longleftrightarrow \quad \begin{cases} x_1 = x_5 \\ x_2 = x_1 \\ x_3 = \overline{x_2} \\ x_4 = x_3 \\ x_5 = x_4 \end{cases}$$

# Fixed points for a negative cycle



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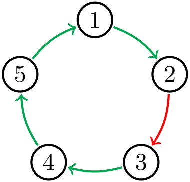
$$f_3(x) = \overline{x_2}$$
$$f_4(x) = x_3$$
$$f_5(x) = x_4$$

$$x = f(x) \iff \begin{cases} x_1 = x_5 \\ x_2 = x_1 \\ x_3 = \overline{x_2} = \overline{x_1} \\ x_4 = x_3 \\ x_5 = x_4 \end{cases}$$

# Fixed points for a negative cycle



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$$f_4(x) = x_3$$
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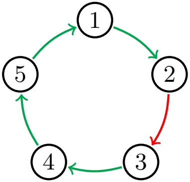
$$x = f(x) \quad \Longleftrightarrow \quad \begin{cases} x_1 = x_5 \\ x_2 = x_1 \\ x_3 = \overline{x_2} = \overline{x_1} \\ x_4 = x_3 = \overline{x_1} \\ x_5 = x_4 \end{cases}$$

# Fixed points for a negative cycle



$$f_1(x) = x_5$$
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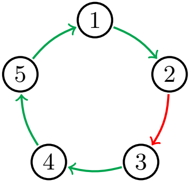

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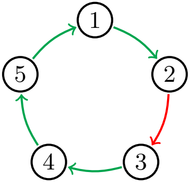

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**There is no fixed point !**

**Proposition**

1. *If $G$ is a positive cycle,*

$$\min(G) = \max(G) = 2.$$

1. *If $G$ is a negative cycle,*

$$\min(G) = \max(G) = 0.$$

## Outline

1. Absence of cycle
2. Positive and negative cycles
3. **Absence of positive/negative cycle**
4. Positive feedback bound
5. Positive and negative cliques
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**Theorem** [Aracena, 2008]

*Let $G$ be an interaction graph.*

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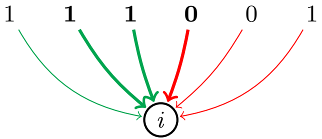
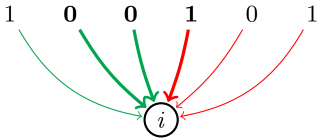
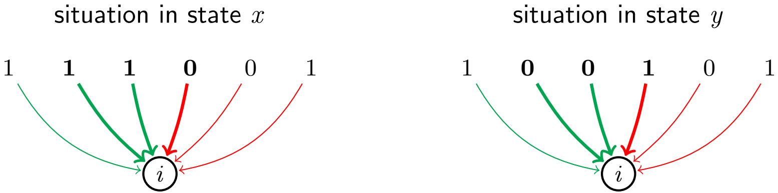
# LOCAL LEMMA


situation in state $x$

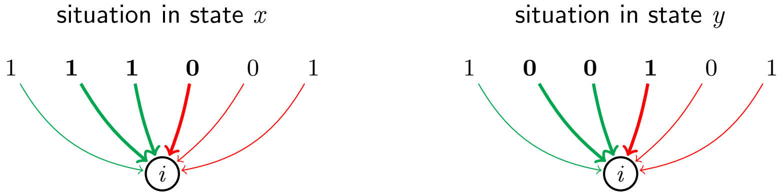

situation in state $y$

## LOCAL LEMMA



**Question :** Can we compare $f_i(x)$ et $f_i(y)$ ?

## LOCAL LEMMA



**Question :** Can we compare $f_i(x)$ et $f_i(y)$ ?

**Réponse :** Yes ! We have $\boldsymbol{f_i(x) \geq f_i(y)}$.

**Theorem** [Aracena, 2008] *If $G$ has only negative cycles, then $\max(G) \leq 1$.*

**Proof**. Let $f$ be a BN on $G$ and let $x$ and $y$ be distinct fixed points of $f$.

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$$\left.\begin{array}{l} x_j \geq y_j \text{ for all } j \to i \\ x_j \leq y_j \text{ for all } j \to i \end{array}\right\} \Rightarrow f_i(x) \geq f_i(y) \Rightarrow x_i \geq y_i \Rightarrow <>$$

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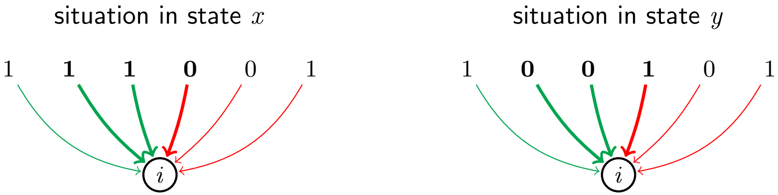

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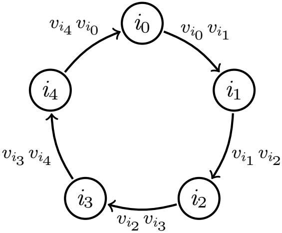
3. There is a cycle $i_0 i_1 i_2 \dots i_\ell i_0$ where the sign of $i_k \to i_{k+1}$ is $v_{i_k} v_{i_{k+1}}$.



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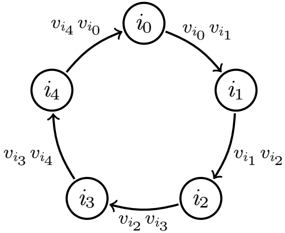

4. The sign $s$ of this cycle is $s = (v_0 v_1) \cdot (v_1 v_2) \cdot (v_2 v_3) \cdot \ \dots \ (\cdot v_\ell v_0) = 1$.

$\square$

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For all $x, y \in \{0,1\}^n$, we set $\Delta(x,y) := \{i \in [n] : x_i \neq y_i\}$.

**Positive cycle lemma.** *If $x$ and $y$ are distinct fixed points of $f$, then*

*$G[\Delta(x,y)]$ has a positive cycle.*

**Question :** Is it difficult to decide if $G$ has a positive/negative cycle ?

$\hookrightarrow$ Reduction to the strongly connected case

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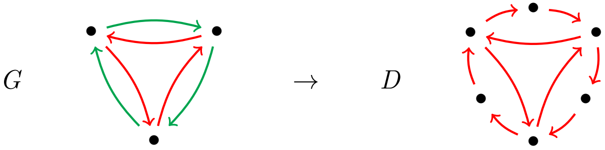

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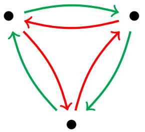
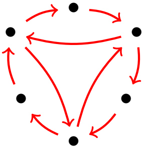
$G$  $\rightarrow$ $D$ 

$G$ has a positive cycle $\iff$ $D$ has an even cycle

$G$ has a negative cycle $\iff$ $D$ has an odd cycle

$\iff$ $D$ is not bipartite

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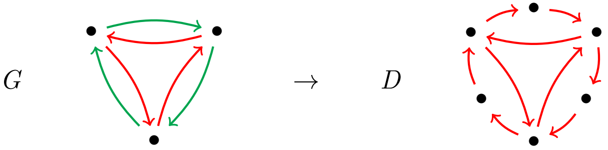

$G$ has a positive cycle $\iff$ $D$ has an even cycle

$G$ has a negative cycle $\iff$ $D$ has an odd cycle

$\iff$ $D$ is not bipartite $O(n^2)$

We can decide in $O(n^2)$ if $D$ is bipartite :

1. We take a spanning tree $T \subseteq D$, and a proper $2$-coloring $c$ of $T$.
2. $D$ is bipartite $\iff$ $c$ is a proper coloring of $D$.

**Question :** Is it difficult to decide if $G$ has a positive/negative cycle ?

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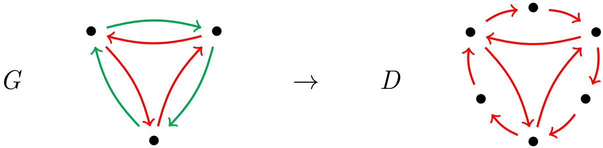

$G$ has a positive cycle $\iff$ $D$ has an even cycle $O(n^d)$

$G$ has a negative cycle $\iff$ $D$ has an odd cycle

$\iff$ $D$ is not bipartite $O(n^2)$

**Theorem** [Robertson-Seymour-Thomas, 1999 ; McCuaig 2004]

*We can decide in polynomial time if $D$ has an even cycle.*

## Outline

1. Absence of cycle
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**We have seen that**

$$G \text{ acyclic } \Rightarrow G \text{ without positive cycle } \Rightarrow \max(G) \leq 1$$

**Do we have something of the form**

$$G \text{ is not so far from being acyclic } \Rightarrow \max(G) \text{ is not too large ?}$$

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**How define a distance to acyclicity ?**

$\hookrightarrow$ *number of cycles ?*

$\hookrightarrow$ *min bn of vertices to delete*
*to make the graph acyclic ?*

$$
\begin{aligned}
\tau(G) &:= \textbf{transversal number}\\
&:= \text{min size of a set of vertices intersecting every cycle}\\
&:= \text{minimum size of a Feedback Vertex Set (FVS)}
\end{aligned}
$$

$\boldsymbol{\tau(G)}$ := **transversal number**

:= min size of a set of vertices intersecting every cycle

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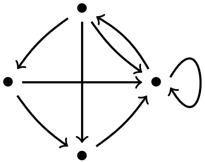
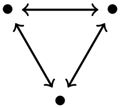
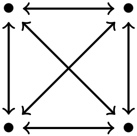


$\tau = 1$



$\tau = 2$



$\tau = 3$

$\boldsymbol{\tau(G)}$ := **transversal number**

:= min size of a set of vertices intersecting every cycle

:= minimum size of a Feedback Vertex Set (FVS)

$\boldsymbol{\tau_p(G)}$ := **positive transversal number**

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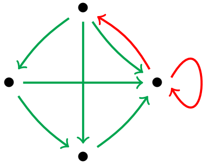
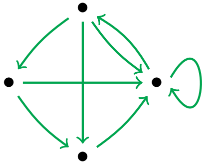


$\tau = 1$

$\tau_p = 0$



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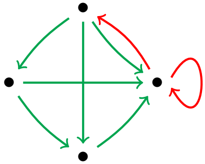
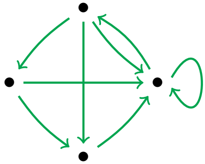
:= min size of a set of vertices intersecting every **positive cycle**



$\tau = 1$
$\tau_p = 0$



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**Remark 1** $\tau_p \leq \tau$ (equality when all arcs are positive)

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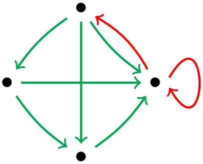

$\tau = 1$
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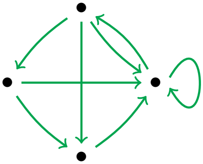

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**Remark 1** $\tau_p \leq \tau$ (equality when all arcs are positive)

**Remark 2** $\tau$ and $\tau_p$ are invariant by subdivisions of arcs

**Positive Feedback Bound** [Aracena 2008]

$$\max(G) \leq 2^{\tau_p} \leq 2^{\tau}$$

**Positive Feedback Bound** [Aracena 2008]

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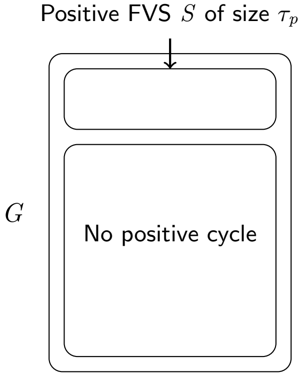

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$G$

No positive cycle

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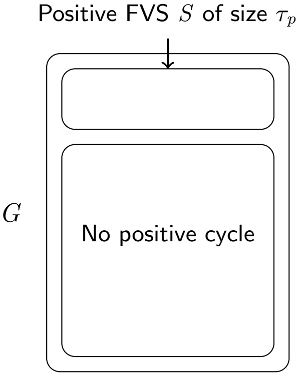
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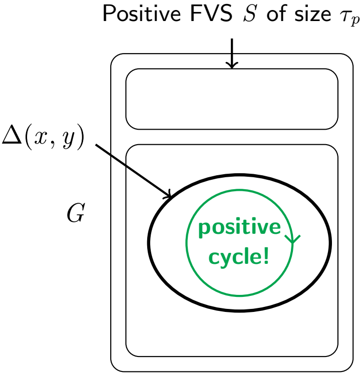

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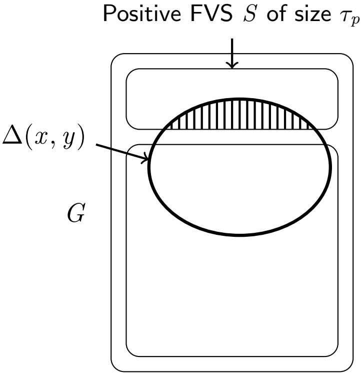

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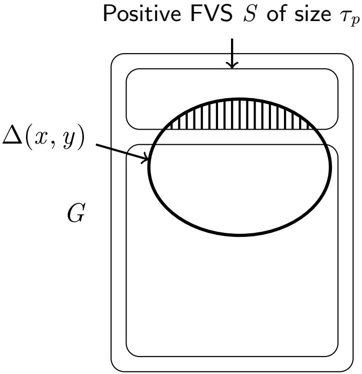

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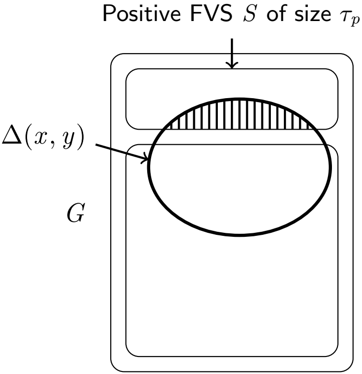

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$$\max(G) \leq 2^{\tau_p} \leq 2^{\tau}$$

**Remark** $G$ has no positive cycle $\Rightarrow$ $\tau_p = 0$ $\Rightarrow$ $\max(G) \leq 1$

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$$\max(G) \leq 2^{\tau_p} \leq 2^{\tau}$$

**Remark** $G$ has no positive cycle $\Rightarrow$ $\tau_p = 0$ $\Rightarrow$ $\max(G) \leq 1$

**This is the only upper bound on $\max(G)$ that only depend on the cycle structure**

*No lower bound on $\max(G)$ !*

**Theorem** [Aracena, 2008]

*Let $G$ be an interaction graph.*

1. *If $G$ has only positive cycles, then $\min(G) \geq 1$.*
2. *If $G$ has only negative cycles, then $\max(G) \leq 1$.*
3. *More generally, $\boldsymbol{\max(G) \leq 2^{\tau_p}}$.*

*Let $G$ be a* **strongly connected** *interaction graph.*

4. *If $G$ has only positive cycles, then $\min(G) \geq 2$.*
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## Remarks

- No general lower bound on $\max(G)$.
- Few results on $\min(G)$.

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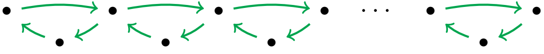
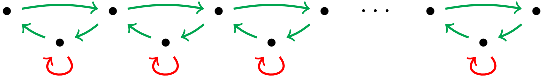
**Theorem** [Bridoux-Durbec-Perrot-R., 2019]

1. *It is* **polynomial** *to decide if $\max(G) \geq 1$.*
2. *It is* **NP-complete** *to decide if $\max(G) \geq 2$.*
3. *It is* **NEXPTIME-complete** *to decide if $\min(G) = 0$.*

# The bound $2^{\tau_p}$ is very perfectible



$$\max(G) \sim 2^{n/6}$$
$$2^{\tau_p} \sim 2^{n/4}$$



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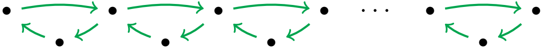

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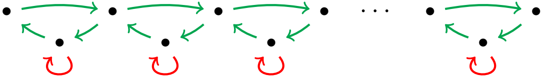

$$\max(G) = 1$$
$$2^{\tau_p} \sim 2^{n/4}$$

How introduce **negative cycles** in the bound ?

↪ Difficult problem : positive cycles are sometime favorable
... and sometime unfavorable to the presence of many fixed points.

The bound $2^{\tau_p}$ is very perfectible

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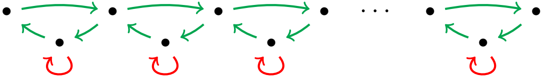

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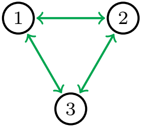
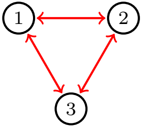


$\max(K_3^+) = 2$



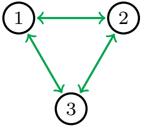
$\max(K_3^-) = 3$

**Two approaches :**

1. Fixe the graph and make variations on signs $\rightarrow$ clique $K_n$.
2. Fixe the signs and make variations of the graphs $\rightarrow$ all arcs positive.

# Outline

1. Absence of cycle
2. Positive and negative cycles
3. Absence of positive/negative cycle
4. Positive feedback bound
5. **Positive and negative cliques**
6. The monotone case
7. Conclusion



8 BNs

$\max(K_3^+) = 2$

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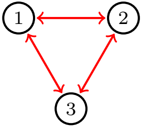

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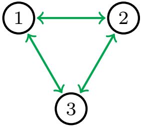

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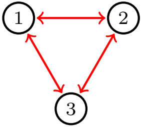

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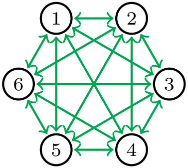
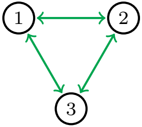


$\sim 10^{41}$ BNs

$$4 \leq \max(K_6^+) \leq 16$$

$$2^{\tau_p} = 2^5 = 32$$



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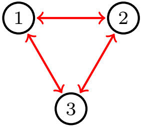

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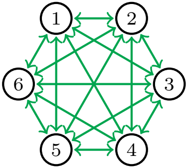

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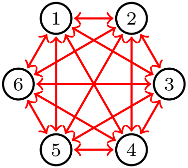

$\sim 10^{41}$ BNs

$$\max(K_6^-) = 20$$

$$2^{\tau_p} = 2^5 = 32$$

## Definitions

1. The **Hamming distance** between two states $x, y \in \{0,1\}^n$ is

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**Example**

$$\begin{array}{rcl} x & = & \mathbf{00}1100\mathbf{11} \\ y & = & \mathbf{11}1100\mathbf{00} \end{array} \qquad d_H(x,y) = 4.$$

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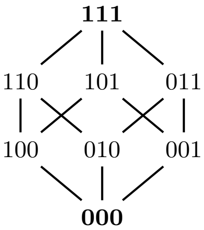

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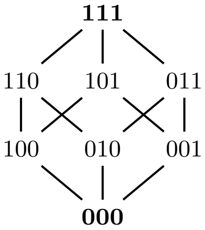

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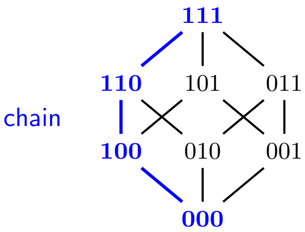

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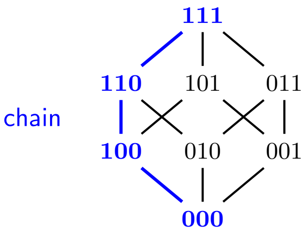

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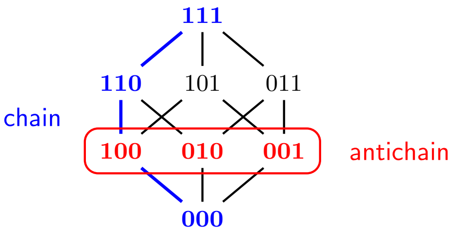

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**Varshamov Bound**

**Sperner Lemma**

**Graham-Sloane Bound**

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**Remark :** In both cases, the positive feedback bound is $2^{n-1}$, while

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = O(2^{n-\log n}) = o(2^{n-1})$$

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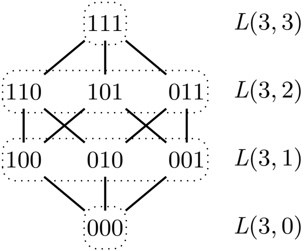
Let $L(n,k)$ the set of $x \in \{0,1\}^n$ with exactly $k$ ones ; $|L(n,k)| = \binom{n}{k}$.



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**Graham-Sloane Bound** [1980]

*It exists* $A \subseteq L(n,k)$ *with* $d_H(x,y) \geq 4$ *for all distinct* $x, y \in A$ *such that*

$$|A| \geq \frac{\binom{n}{k}}{n}.$$

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Let $f$ be a BN on $K_n^+$. If $x$ and $y$ are distinct fixed points of $f$, then

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**Varshamov Bound** [1965]

*If* $A \subseteq \{0,1\}^n$ *and* $d_{\max}(x,y) \geq 2$ *for all distinct* $x, y \in A$ *distincts, then*

$$|A| \leq \frac{2^{n+1}}{n+2}.$$

**Theorem** [Gadouleau-R-Riis, 2015]

$$\frac{\binom{n}{\lfloor \frac{n}{2} \rfloor}}{n} \leq \max(K_n^+) \leq \frac{2^{n+1}}{n+2} \leq \max(K_n^-) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

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Let $f$ be a BN on $K_n^-$. we have $x \leq y \Rightarrow f(x) \geq f(y)$.

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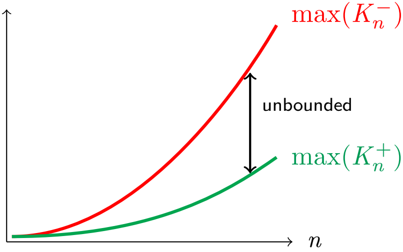

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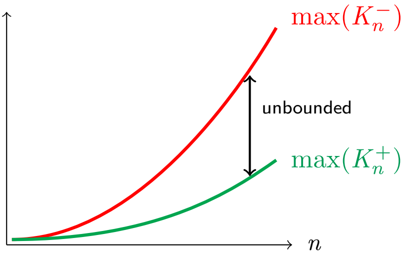

**Corollary.** *For all fixed $k$ and sufficiently large $n$,*

$$\max(K_n^-) > \max(K_{n+k}^+).$$

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**Conjecture**

*If $K_n^\sigma$ is a signed clique with $n$ vertices,*

$$\max(K_n^\sigma) \leq \max(K_n^-)$$

**Two approaches :**

1. Fixe the graph and make variations on signs → clique $K_n$.
2. Fixe the signs and make variations of the graphs → all arcs positive.

## Outline

1. Absence of cycle
2. Positive and negative cycles
3. Absence of positive/negative cycle
4. Positive feedback bound
5. Positive and negative cliques
6. **The monotone case**
7. Conclusion

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**Proposition**

1. *Suppose that $G$ is strongly connected and has only positive cycles. Let $G^+$ be obtained from $G$ by making positive every arc. Then*

$$\max(G) = \max(G^+).$$

2. *Furthermore, every BN $f$ on $G^+$ is* **monotone***, that is,*

$$\forall x, y \in \{0,1\}^n \qquad x \leq y \;\Rightarrow\; f(x) \leq f(y).$$

**Theorem** [Knaster-Tarski, 1928]

*If $f$ is monotone then* $\mathsf{Fix}(f)$ *is a non-empty lattice. In particular, $f$ has a unique minimal fixed point and a unique maximal fixed point (wrt $\leq$).*

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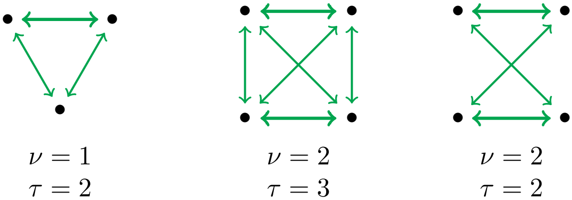

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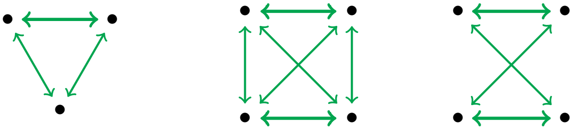

$\nu = 1$
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**Remark** $\nu \leq \tau$

**Theorem** [Aracena-Salinas-R, 2017]

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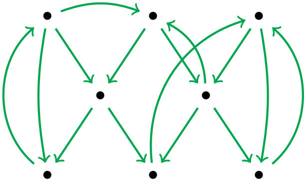

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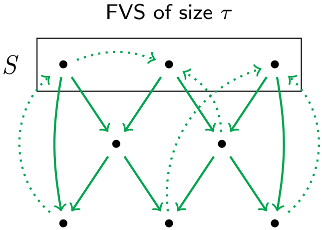

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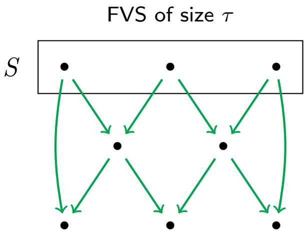

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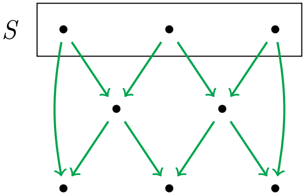
FVS of size $\tau$



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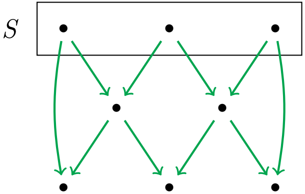

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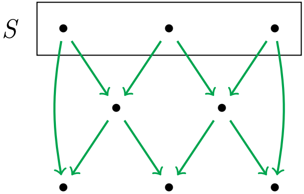

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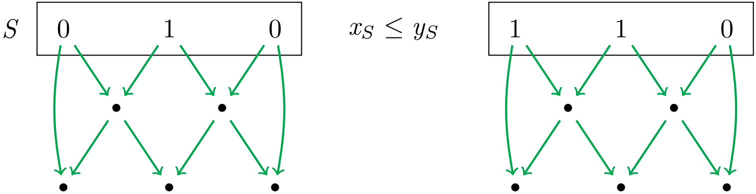

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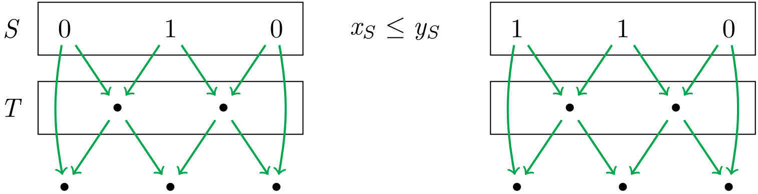

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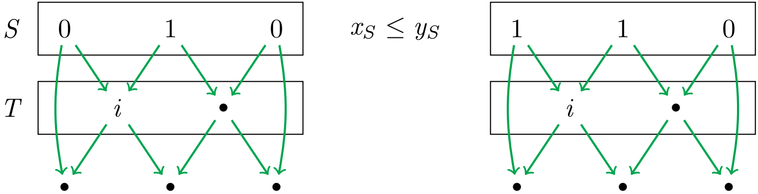

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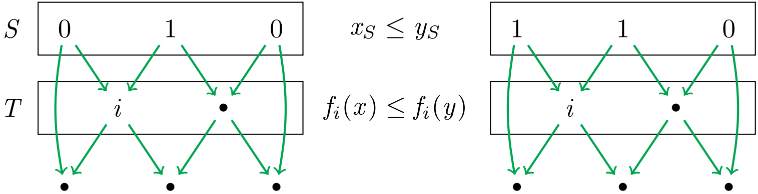

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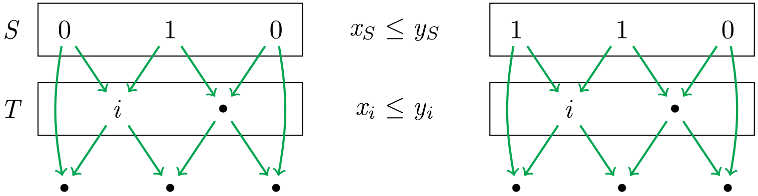

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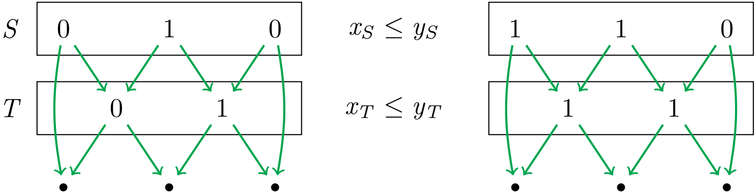

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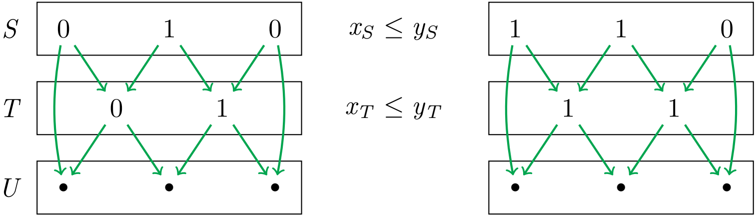

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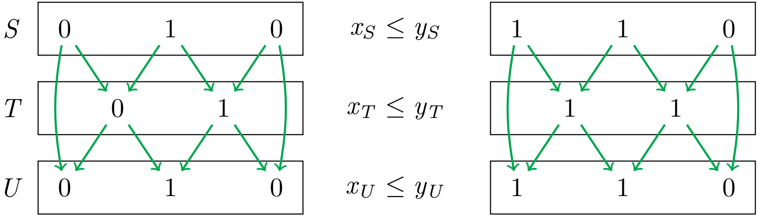

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 & \multicolumn{4}{c}{\Delta(x^1,x^2)} & \multicolumn{5}{c}{\Delta(x^2,x^3)} & \multicolumn{4}{c}{\Delta(x^3,x^4)} & \multicolumn{5}{c}{\Delta(x^4,x^5)}
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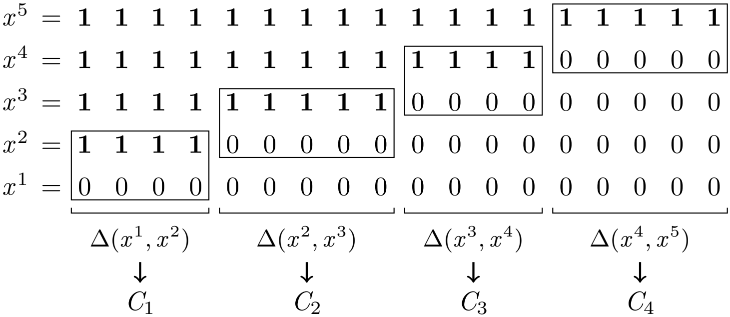

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x^2 & = & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x^1 & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}$$

$$\underbrace{\quad\quad}_{\Delta(x^1,x^2)} \qquad \underbrace{\quad\quad}_{\Delta(x^2,x^3)} \qquad \underbrace{\quad\quad}_{\Delta(x^3,x^4)} \qquad \underbrace{\quad\quad}_{\Delta(x^4,x^5)}$$

$$\Delta(x^1,x^2) \downarrow C_1 \qquad \Delta(x^2,x^3) \downarrow C_2 \qquad \Delta(x^3,x^4) \downarrow C_3 \qquad \Delta(x^4,x^5) \downarrow C_4$$

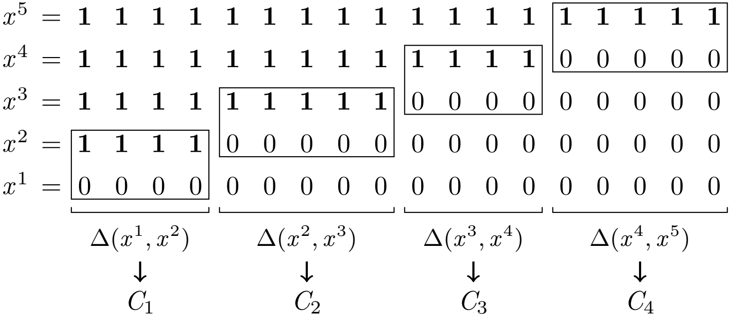
**Theorem** [Aracena-Salinas-R, 2017]

*If $f$ is monotone,* $\mathrm{Fix}(f)$ *is isomorphic to a subset $L \subseteq \{0,1\}^\tau$ such that*

1. *$L$ is a non-empty lattice*
2. *$L$ has no chain of size $\nu + 2$*

**Proof of 2** *If* $\mathrm{Fix}(f)$ *has a chain of $k+1$ fixed points then $\nu \geq k$.*

Thus $\mathrm{Fix}(f)$ has no chain of size $\nu + 2$, and $L$ also.



$$\begin{array}{rcccccccccccccccccccc}
x^5 &=& \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\
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**Theorem** [Erdős, 1945]

*If* $A \subseteq \{0,1\}^n$ *has no chain of size* $\ell + 1$ *then*

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Remark The case $\ell = 1$ is Sperner Lemma on antichains

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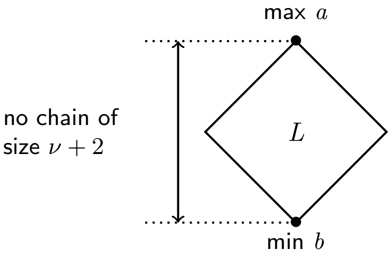

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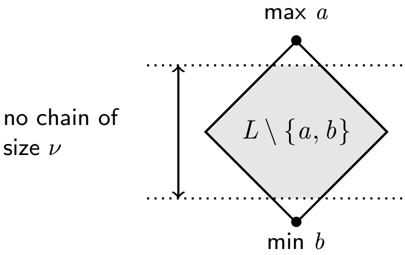

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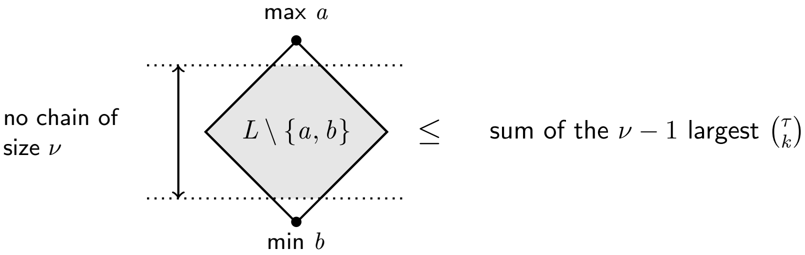

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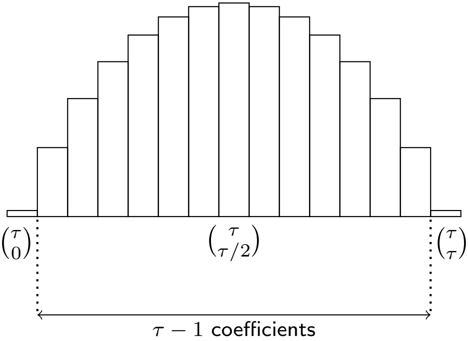

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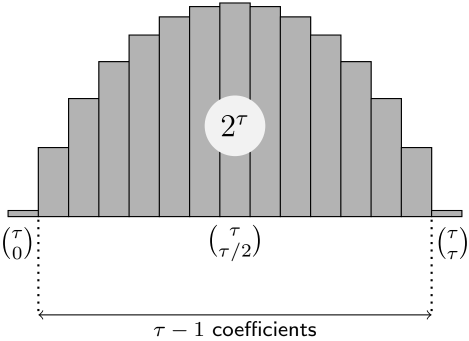

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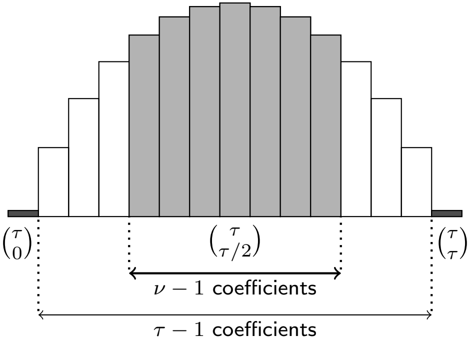

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**Conjecture** $\max(G) \leq 2^{O(\nu \log \nu)}$

## Outline

1. Absence of cycle
2. Positive and negative cycles
3. Absence of positive/negative cycle
4. Positive feedback bound
5. Positive and negative cliques
6. The monotone case
7. **Conclusion**

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**Graphe Theory**

Even/odd cycles
Erdős-Pósa property

**Set Theory**

Sperner Lemma
Erdős extension
Tarski Theorem

**Coding Theory**

Graham-Sloane bound
Varshamov bound

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**Conjecture :** $\max(G)$ can be bounded according to the maximum number of vertex-disjoint positive cycles in $G$.

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**Gracias !**