Introduction to the Boolean networks: motivation and basic concepts

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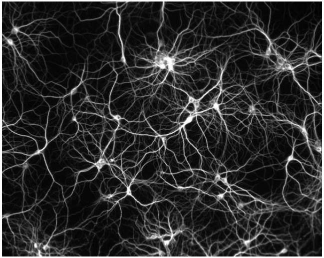
Complex systems

Examples of complex dynamics networks:



Neural network
(Image from Universite Laval)

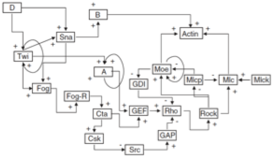
Social network
(Image from University of Oxford)

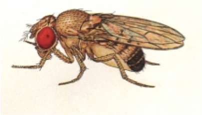
Boolean networks in biology

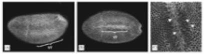
Boolean networks (BNs) were introduced by Stuart Kauffman in 1969 to model the gene regulatory networks which consists of a set of genes, proteins, small molecules, and their mutual interactions.

Representation:

- **Vertex** = A gene or a gene product.
- **States** = 1 (activated), 0 (inactivated).
- **Interaction Graph** = Interaction of genes and genes products each other.
- **Activation function** = Regulation function.
- **Updating** = parallel (in the most cases).
- **Attractors** = Cellular phenotypes and cycles.







(Aracena J. et al. Journal of Theoretical Biology, 2006.)

A Boolean network is a system of a set of n elements interacting, where each one of them has a Boolean variable x_i associated which evolve, in a discrete time, according to a predefined rule.

Many applications

- Neural networks [McCulloch & Pitts 1943]
- Gene networks [Kauffman 1969, Tomas 1973]
- Epidemic diffusion, social network, etc.

A **Boolean network** (BN) of n components is a function

$$f : \{0,1\}^n \rightarrow \{0,1\}^n$$
$$x = (x_1, \ldots, x_n) \mapsto f(x) = (f_1(x), \ldots, f_n(x))$$

The functions $f_i : \{0,1\}^n \rightarrow \{0,1\}$ are called local activaction functions.

The dynamics of a BN f (under parallel schedule) is described by its successive iterations:

$$x \rightarrow f(x) \rightarrow f^2(x) \rightarrow f^3(x) \rightarrow \cdots$$

or equivalently:

$$\forall t \in \mathbb{N}_0, \forall x(t) \in \{0,1\}^n : x(t+1) = f(x(t)).$$

The dynamics of a BN $f : \{0,1\}^n \rightarrow \{0,1\}^n$ can be represented by a digraph, named iteration graph, defined as:

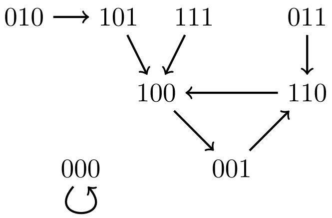
$$G_I(f) = (V = \{0,1\}^n, A = \{(x, f(x)) : x \in V\})$$

Example with $n = 3$ and f defined by

$$\begin{cases} f_1(x) &= x_2 \vee x_3 \\ f_2(x) &= \overline{x}_1 \wedge x_3 \\ f_3(x) &= \overline{x}_3 \wedge (x_1 \oplus x_2) \end{cases}$$

x	$f(x)$
000	000
001	110
010	101
011	110
100	001
101	100
110	100
111	100

Dynamics: G_I



Dynamical characteristics

Given f a BN of n components and $x \in \{0,1\}^n$ a configuration, we say that:

1. x is a Garden of Eden if $f^{-1}(\{x\}) = \emptyset$.
2. x is a fixed point if $f(x) = x$.
3. x is a periodic point if $\exists k \in \mathbb{N}, f^k(x) = x$; x is transient otherwise.
4. A limit cycle is a cycle of $G_I(f)$ of length at least two.
5. An attractor is a terminal strongly connected component of $G_I(f)$ (ie limit cycle or fixed point). Basin of attraction: set of states reaching an attractor.
6. The period of f, denoted by $p(f)$, is the least common multiple of all lengths of its limit cycles.
7. The height of f, denoted by $h(f)$, is the least k such that $\forall x \in \{0,1\}^n,\ f^k(x)$ is a periodic point. Hence,

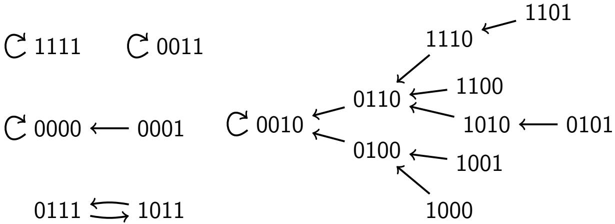
$$f^{h+p} = f^h,$$

where $h = h(f)$ and $p = p(f)$.

Example with $n = 4$ and f defined by

$$\begin{cases} f_1(x) &= x_2 \wedge x_4, \quad f_3(x) &= x_2 \vee x_3 \\ f_2(x) &= x_1, \quad f_4(x) &= x_3 \wedge x_4 \end{cases}$$

Dynamics: G_I



Gardens of Eden: $0001, 1101, 1100, 0101, 1001, 1000.$
Fixed points: $0000, 0010, 0011, 1111.$
Limit cycle: $[0111, 1011, 0111]$; $p(f) = 2$; $h(f) = 3.$

The **interaction graph** of f of n components is the digraph $G(f)$ defined by

- the vertex set is $[n] := \{1, \dots, n\}$
- there is an arc $j \to i$ if f_i *depends on* x_j

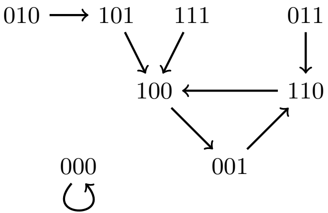
We say that a function $f_i : \{0,1\}^n \to \{0,1\}$ depends on x_j if $\exists x \in \{0,1\}^n$ such that

$$f_i(x_1, \dots, x_j, \dots, x_n) \neq f_i(x_1, \dots, \overline{x}_j, \dots, x_n).$$

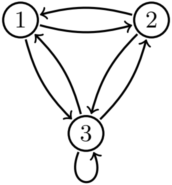
Example with $n = 3$ and f defined by

$$\begin{cases} f_1(x) &= (x_2 \wedge \overline{x}_3) \vee x_3 = x_2 \vee x_3 \\ f_2(x) &= \overline{x_1} \wedge \overline{x_3} \\ f_3(x) &= \overline{x_3} \wedge (x_1 \oplus x_2) \end{cases}$$

Dynamics



Interaction graph



Some natural questions:

- *How can we infer a Boolean network from a set of observations?*
- *What can be said on the dynamics of a system according to its interaction graph ?*
- *What can be said on the robustness of the dynamics a Boolean network against to changes in its parameters?*

Gracias!